Grade 9/10 Math Circles
April 3, 2024
Probability III - Solutions

In-Lesson Exercises

1. The number of digits in the PIN was not used in our calculation for the previous example, so the probability is the same! This is because the choice for the other digits is independent of the choice for the first and last digit.

2. Let C be the event that you flip heads exactly twice and B be the event that you flip heads at least once. Notice that C is a subset of B - if you flip heads exactly two times then you certainly flipped heads at least once. So, $C \cap B = C$.

   There are 8 possible outcomes of flipping a coin 3 times, all equally likely.

   There are 3 ways to flip exactly two heads (HHT, HTH, THH), so $P(C) = 3/8$.

   From the previous example, we know $P(B) = 7/8$.

   Putting this together with the definition of conditional probability,

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C)}{P(B)} = \frac{3/8}{7/8} = \frac{3}{7}$$

   Perhaps surprisingly, this is the same probability as flipping heads exactly once!

3. Let M mean a person is a man and C mean a person is colour-blind.

   (a) The probability of being colour-blind depends on whether someone is a man. So, we split the probability of being colour-blind as follows.

$$P(C) = P(C|M) \cdot P(M) + P\left(C|M^C\right) \cdot P\left(M^C\right) = \frac{1}{12} \cdot \frac{1}{2} + \frac{1}{200} \cdot \frac{1}{2} = \frac{53}{1200}$$

   (b) We use Bayes' Theorem to find $P(M|C)$.

$$P(M|C) = \frac{P(C|M) \cdot P(M)}{P(C)} = \frac{\frac{1}{12} \cdot \frac{1}{2}}{\frac{53}{1200}} = \frac{50}{53}$$

4. As before, we use Bayes' Theorem. This time, we must calculate

$$P(H|P) = \frac{P(P|H) \cdot P(H)}{P(P)}$$

We know that $P(P|H) = 0.01$ and $P(H) = 0.98$.

To find the probability of a positive test, we again need to consider the two cases: you are sick with COVID or you are not sick with COVID.

$$P(P) = P(P|S) \cdot P(S) + P(P|H) \cdot P(H)$$

The only value we don't know is the probability of an accurate positive, $P|S$. A sick person who takes a COVID test will either get a positive or negative result, so $P(P|S) = 1 - P(N|S) = 0.9$.

Putting this all together gives:

$$\begin{aligned} P(H|P) &= \frac{P(P|H) \cdot P(H)}{P(P)} \\ &= \frac{P(P|H) \cdot P(H)}{P(P|S) \cdot P(S) + P(P|H) \cdot P(H)} \\ &= \frac{0.01 \cdot 0.98}{0.9 \cdot 0.02 + 0.01 \cdot 0.98} \\ &\approx 0.35 \end{aligned}$$

So, a positive COVID test still gives you a 35% chance of being healthy! This is because most people who take a test are not sick, so a large amount of the total positive events are going to be due to false positives.

5. (a) This is a random variable, which takes values between 0 and 4.
   (b) This is not a random variable, as it does not take on numerical values.
   (c) This is a random variable, which takes values between 0 and the number of games each player competes in.

6. Let X be the number of heads from flipping 2 coins. Using the values from example 7,

$$\begin{aligned} E[X] &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\ &= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} \\ &= 0 + \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

7. Let X be the value of the roll from your perspective.

   There is a 2/3 chance the die lands on 1 through 4 and a 1/3 chance it lands on a 5 or 6.

$$\begin{aligned} E[X] &= 3 \cdot P(X=3) - 7 \cdot P(X=-7) \\ &= 3 \cdot \frac{2}{3} - 7 \cdot \frac{1}{3} \\ &= \frac{6}{3} - \frac{7}{3} = -\frac{1}{3} \end{aligned}$$

   So, you would expect to lose money and should not play the game.

   To make it fair, we want $E[X] = 0$, so we must solve

$$\begin{aligned} 0 &= 3 \cdot P(X=3) - k \cdot P(X=-k) \\ &= 3 \cdot \frac{2}{3} - k \cdot \frac{1}{3} \\ &= \frac{6}{3} - \frac{k}{3} = \frac{6-k}{3} \end{aligned}$$

   So, you should set $k = 6$ to make the game fair.